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Angular momentum fluctuations of the ideal Bose gas in a rotating bucket

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Abstract. An equilibrium superfluid is usually modelled by a Bose gas below its transition temperature. Here, we compute rigorously the extremal equilibrium states below this transition temperature. We compute the deviation from normality of the angular momentum fluctuations both above and below this transition temperature. Finally, we compute the distribution functions of the angular momentum and its fluctuations.

1. Introduction

Blatt and Butler [1] have shown that a rotating ideal Bose gas in three dimensions undergoes phase transitions similar to those occurring in rotating He II. Their main result is that the total angular momentum J of the gas, considered as a function of the angular velocity ω of the bucket, increases linearly between a sequence $\omega_1, \omega_2, \ldots$ of critical values of ω. At a critical value of ω, the angular momentum jumps by an amount $N_0\hbar$, where N_0 is the number of condensed particles.

Blatt *et al* [2] considered the rotating bucket model and discussed the question as to whether the effective moment of inertia is the same for classical as well as for quantum statistical mechanics. By heuristic computations, they showed that this holds above the transition point, but not below.

Lewis and Pulé [3] made a rigorous study of the free Bose gas in a rotating bucket. They treated a grand canonical Bose gas by fixing the average density and the average angular momentum. The idea is to compute the generating functional of the representation of the cyclic representation of the canonical commutation relations, corresponding to the state which is the thermodynamic limit of the grand canonical ensemble with fixed density and fixed angular momentum. Using this state, they showed that there exists a critical density ρ_c above which there is a condensate in the lowest, or the two lowest, energy levels, depending on the angular velocity of the system in the thermodynamic limit. These computations confirmed the heuristic results of Blatt and Butler [1] and Putterman *et al* [4].

In section 2, we recall the main properties of the model of the rotating bucket as described in [3]. We need two of their results. We extend their first in the sense that, for $\bar{\rho} > \rho_c$, we calculate the extremal equilibrium states. In order to achieve this, we add one or two extra field terms to the Hamiltonian, a technique already used in [5] and [6]. We also indicate different, but complementary, ways of calculating this equilibrium state. Loosely speaking, one can say that we calculate with test functions, which feel the boundary. In [3],

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the state is calculated for functions which do not see the boundary. The precise statement can be found in section 2.

Section 3 contains essentially the law of large numbers. We prove that for densities below the critical density ρ_c, the distribution functions of the average angular momentum density and the average particle density are given by a δ-distribution. For densities higher than the critical density however, we rigorously prove that the distribution function of the angular momentum corresponding to the state given in [3] is certainly not a δ-distribution. However, the distribution function of the angular momentum corresponding to the extremal state is a δ-distribution. This confirms the fact that the states which we derived are extremal equilibrium states.

In section 4, we introduce the fluctuation of the angular momentum and make precise in what sense we take the thermodynamic limit. Since the fluctuation of the angular momentum is directly related to the *moment of inertia* of the system, one expects the fluctuations to behave as $O(L^5)$. Our computations show indeed this behaviour if $\bar{\rho} < \rho_c$, which is in agreement with the heuristic computations of Blatt *et al* [2]. For $\bar{\rho} > \rho_c$, we make the external field volume dependent in the sense that it tends to zero with increasing volume, i.e. we treat it as a boundary condition and we compute the deviation from normality. This is expressed in terms of a parameter which we call the *critical exponent*. If the field vanishes very slowly, the critical exponent is independent of the rate of decay of the field and coincides with the exponent for the case of low densities. If the field drops off too quickly, the effect of the field terms disappears and the state is no longer extremal. In the case where the field vanishes moderately quickly, the critical exponent depends on the vanishing rate. The results of this section, without proofs, have already been announced in [7].

The final section contains the proof that the distribution function of the moment of inertia of the system is Gaussian. This result is true both below and above the critical density and is independent of the boundary condition.

2. The model

We follow closely the set-up of Lewis and Pulé [3]. Let Λ_1 be the cylindrical region of unit volume in $\mathbb{R}^3$:

$$\Lambda_1 = \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < a^2, |x_3| < (2\pi a^2)^{-1}\right\}$$

and $\forall L > 0$, we define

$$\Lambda_L = \left\{x \in \mathbb{R}^3 : \frac{x}{L} \in \Lambda^1\right\}$$

and let $\mathcal{H}_L = L^2(\Lambda_L)$. Denote by $h_L = \frac{-\Delta}{2} - \Omega_L j_L$ the one-particle Hamiltonian, a self-adjoint operator on $\mathcal{H}_L$, with Neumann boundary condition on $\partial\Lambda_L$: $\partial\phi/\partial n = 0$, where $\partial\phi/\partial n$ is the normal derivative and Δ is the Laplacian. The operator $j_L = -\mathrm{i}\partial/\partial\theta$ represents the angular momentum around the x_3-axis. Ω_L can be interpreted as the angular velocity of the system.

In order to describe bosons in a rotating bucket, one considers the Fock space $\mathcal{F}(\mathcal{H}_L)$. Denote by H_L and J_L the operators on $\mathcal{F}(\mathcal{H}_L)$ induced by h_L and j_L, respectively. The algebra $\mathcal{A}_L$, which describes the system of bosons, is a CCR-algebra, generated by the Fock creation and annihilation operators $a^+(f)$ and $a(g)$, $\forall f, g \in \mathcal{H}_L$ [8]. They satisfy the following relations:

$$a(f)^* = a^+(f) \qquad [a(f), a^+(g)] = \langle f, g\rangle_L$$

$$[a(f), a(g)] = 0 \qquad a(f) = \int_{\Lambda_L} \mathrm{d}x\, f(x)a(x)$$

where $\langle .\,, .\rangle_L$ denotes the scalar product in $\mathcal{H}_L$. The system of free bosons in a rotating bucket is now described by the following Hamiltonian:

$$H_L - \mu_L N_L = -\tfrac{1}{2}\int_{\Lambda_L} \mathrm{d}x\, \nabla a^+(x)\nabla a(x) - \Omega_L J_L - \mu_L N_L \tag{1}$$

where μ_L is the chemical potential and N_L is the number operator on $\mathcal{F}(\mathcal{H}_L)$. A state is a normalized linear functional on the algebra $\mathcal{A}_L$. A state η_L which describes the system in thermal equilibrium satisfies the KMS equation

$$\eta_L(A\alpha_{i\beta}B) = \eta_L(BA) \qquad \forall A, B \in \mathcal{A}_L \tag{2}$$

where $\alpha_{i\beta}A = \mathrm{e}^{-\beta(H_L-\mu_L N_L)}A\mathrm{e}^{\beta(H_L-\mu_L N_L)}$. For finite volumes, the solution of this equation is unique and given by the Gibbs state

$$\eta_L(A) = \frac{\mathrm{Tr}\,\mathrm{e}^{-\beta(H_L-\mu_L N_L)}A}{\mathrm{Tr}\,\mathrm{e}^{-\beta(H_L-\mu_L N_L)}} \tag{3}$$

where $\beta = 1/kT$ is the inverse temperature. Because algebra $\mathcal{A}_L$ is generated by $\{\mathbb{1}, a(f), a^+(g)\}$ with $f, g \in \mathcal{H}_L$, the state is completely determined by the correlation functions

$$\eta_L\left(\prod_{i=1}^{n} a^+(f_i)\prod_{j=1}^{m} a(g_j)\right)$$

for all $m, n \in \mathbb{N}$ and $f_i, g_j \in \mathcal{H}_L$. These are calculated by means of the generating functional, i.e. the expectation value of the Weyl operators:

$$W(f) = \exp\frac{\mathrm{i}}{\sqrt{2}}(a(f) + a^+(f)) \qquad f \in \mathcal{H}_L.$$

It is a well known fact that this expectation value is of the form

$$\eta_L(W(f)) \equiv \mu_{\beta,\Omega_L,\mu_L}(f) = \mu_F(f)\exp -\tfrac{1}{2}\mathcal{A}_{\beta,\mu_L,\Omega_L}(f, f) \tag{4}$$

where

$$\mu_F(f) = \mathrm{e}^{-\frac{1}{4}\|f\|^2} \qquad \mathcal{A}_{\beta,\mu_L,\Omega_L}(f, f) = \left\langle f, \frac{\exp(h_L - \Omega_L j_L)}{\exp(h_L - \Omega_L j_L) - 1} f\right\rangle_L.$$

First we determine a suitable basis in $\mathcal{H}_L$. Let $\{\phi_{k,L} : k = 1, 2, 3, \ldots\}$ be a complete orthonormal set in $\mathcal{H}_L$ such that

$$\begin{cases} -\dfrac{\Delta}{2}\phi_{k,L} = E_{k,L}\phi_{k,L} \\ j_L\phi_{k,L} = l_k\phi_{k,L} \end{cases} \tag{5}$$

where k stands for the three quantum numbers (n, l, m) appropriate to the cylindrical geometry. The labels k are such that

$$E_{k,L} - \Omega_L l_k \leqslant E_{k+1,L} - \Omega_L l_{k+1}.$$

$\phi_{k,L}$ are given in terms of $\phi_k \equiv \phi_{k,1}$ and $E_{k,L}$ in terms of $E_{k,1}$ as follows

$$\phi_{k,L}(x) = L^{-3/2}\phi_k\left(\frac{x}{L}\right)$$

$$E_{k,L} = L^{-2}E_{k,1}$$

where

$$\phi_k = K_{l,n} J_{|l|}\left(\frac{r_{l,n}r}{a}\right) e^{il\theta} \cos\left(\pi^2 a^2 m x_3 + \frac{\pi m}{2}\right)$$

$l = 0, \pm 1, \pm 2, \ldots$, $m = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$ and $r_{l,n}$ is the nth non-negative zero of $J'_{|l|}$ in increasing order. Furthermore,

$$K_{0,0} = 1$$

$$K_{l,n} = \frac{\sqrt{2}r_{l,n}}{2\pi a^2 J_{|l|}(r_{l,n})\sqrt{r_{l,n}^2 - l^2}} \qquad |l| + n \geqslant 1$$

and the set of energies is

$$E_k = \tfrac{1}{2}(a^{-2}r_{l,n}^2 + \pi^4 a^4 m^2).$$

It is straightforward to rewrite the Hamiltonian in this basis:

$$H_L = \sum_{k=1}^{\infty}(E_{k,L} - \Omega_L l_k - \mu_L)a_{k,L}^+ a_{k,L}$$

and

$$J_L = \sum_{k=1}^{\infty} l_k a_{k,L}^+ a_{k,L} \tag{6}$$

$$N_L = \sum_{k=1}^{\infty} a_{k,L}^+ a_{k,L} \tag{7}$$

where $a_{k,L} \equiv a(\phi_{k,L})$. One computes the average particle and angular momentum densities

$$\bar{\rho} \equiv \frac{\eta_L(N_L)}{L^3} = \frac{1}{L^3}\sum_{k=1}^{\infty}\frac{1}{e^{\beta(E_{k,L}-\Omega_L l_k-\mu_L)} - 1} \tag{8}$$

$$\bar{\lambda} \equiv \frac{\eta_L(J_L)}{L^3} = \frac{1}{L^3}\sum_{k=1}^{\infty}\frac{1}{e^{\beta(E_{k,L}-\Omega_L l_k-\mu_L)} - 1}. \tag{9}$$

One takes the limit $L \to \infty$, keeping $\bar{\rho}$ and $\bar{\lambda}$ fixed. Constraints (8) and (9) determine μ_L and Ω_L as functions of L.

We remark here that we follow the same point of view as in [10], where the thermodynamic limit is taken keeping the density constant. Here we keep the density, as well as the angular momentum density, fixed.

On the interval [0, 1], $\forall \alpha > 0$, let

$$g_\alpha(z) \equiv \sum_{n=1}^{\infty} n^{-\alpha} z^n.$$

The function $z \longrightarrow (2\pi\beta)^{-3/2} g_\alpha(z)$ is continuous on [0, 1] and increases monotonically to a maximum ρ_c at $z = 1$, such that for $\bar{\rho} \leqslant \rho_c$, the equation $\bar{\rho} = (2\pi\beta)^{-3/2} g_\alpha(\theta)$ has a unique root $\theta(\bar{\rho})$. We are interested in the limit state $\eta = \lim_{L\to\infty} \eta_L$. It is given by the following theorem [3] (see also equation (4)).

Theorem 1. Let $\mathcal{D}$ be the space of C_0^∞ functions on $\mathbb{R}^3$ having compact support. Then, for each h in $\mathcal{D}$, the quadratic form $\mathcal{A}_{\beta,\bar{\rho},\bar{\lambda}}$ is given on $\mathcal{D}$ by

$$\mathcal{A}_{\beta,\bar{\rho},\bar{\lambda}} = \begin{cases} \langle h, f_{\theta(\rho)} h\rangle & \text{if } \bar{\rho} \leqslant \rho_c \\ (\bar{\rho} - \rho_c) G\left(\dfrac{\bar{\lambda} - \frac{1}{2}\omega_1 a^2 \rho_c}{(\bar{\rho} - \rho_c)}\right) |\hat{h}(0)|^2 + \langle h, f_1 h\rangle & \text{if } \bar{\rho} > \rho_c \end{cases}$$

where

$$f_z(h) = \int_{\mathbb{R}^3} f_z(\| x - y \|) h(y)\, \mathrm{d}y$$

with

$$f_z(s) = (2\pi\beta)^{-3/2} \sum_{n=1}^{\infty} \mathrm{e}^{-s^2/2n\beta} n^{-3/2} z^n$$

and

$$G(t) = \begin{cases} 1 & t < 0 \\ 1 - t & 0 \leqslant t \leqslant 1 \\ 0 & t > 1 \end{cases}$$

$$\omega_l = \frac{1}{2a^2}(r_{l,1}^2 - r_{l-1,1}^2) \qquad \text{for } l \geqslant 1.$$

For $\bar{\rho} \leqslant \rho_c$, this state is an extremal equilibrium state in the sense that it cannot be decomposed into a non-trivial convex combination of two other states η_1, η_2 that are solutions of (2), i.e.

$$\eta \neq \lambda\eta_1 + (1 - \lambda)\eta_2 \qquad \text{with} \qquad \lambda \in]0, 1[\qquad \text{and} \qquad \eta_1 \neq \eta_2. \tag{10}$$

However, if $\bar{\rho} > \rho_c$, the state is no longer extremal (see section 4). In order to calculate the extremal states, one adds one or two extra field terms to the Hamiltonian.

Before doing this, we recall another result [3]. Denote $n_{i,L} = a_{i,L}^+ a_{i,L}$ and define

$$a_L(\Omega, \mu) \equiv \frac{\eta_L(n_{1,L}) + \eta_L(n_{2,L})}{L^3} = \frac{1}{L^3} \sum_{i=1}^{2} \frac{1}{\mathrm{e}^{\beta(E_{i,L} - \Omega_L l_i - \mu_L)} - 1} \tag{11}$$

$$c_L(\Omega, \mu) \equiv \frac{\eta_L(n_{2,L})}{L^3} = \frac{1}{L^3} \frac{1}{\mathrm{e}^{\beta(E_{2,L} - \Omega_L l_2 - \mu_L)} - 1}. \tag{12}$$

These are the lowest energy-level contributions to the average density. Note that because of stability, we have $E_{1,L} - \Omega_L l_1 - \mu_L \geqslant 0$. Denote $\omega(L) = L^2 \Omega_L$, then we have the following theorem [9].

Theorem 2. If $\bar{\rho} \leqslant \rho_c$, then

$$a_L(\omega, \mu) \to 0 \qquad \omega(L) \to \frac{2\bar{\lambda}}{a^2\bar{\rho}} \qquad e^{-\beta(E_1^L - \Omega_L l_1 - \mu_L)} \to \theta(\bar{\rho})$$

where $\theta(\bar{\rho})$ is the root of the equation

$$\bar{\rho} = (2\pi\beta)^{-3/2} g_{3/2}(\theta). \tag{13}$$

If $\bar{\rho} > \rho_c$, then

$$a_L(\omega, \mu) \to \bar{\rho} - \rho_c \qquad e^{-\beta(E_1^L - \Omega_L l_1 - \mu_L)} \to 1.$$

(i) If, for some $l \geqslant 0$,

$$l(\bar{\rho} - \rho_c) + \frac{\omega_l a^2 \rho_c}{2} < \bar{\lambda} < l(\bar{\rho} - \rho_c) + \frac{\omega_{l+1} a^2 \rho_c}{2}$$

with ω_l as defined in theorem 1, then there exists a $\omega' \in]\omega_l, \omega_{l+1}[$ such that

$$\bar{\lambda} = l(\bar{\rho} - \rho_c) + \frac{\omega' a^2 \rho_c}{2} \qquad \omega(L) \to \omega' \qquad l_1(\omega(L)) = l \qquad c_L(\omega, \mu) \to 0. \tag{14}$$

(ii) If, for some $l \geqslant 1$,

$$(l - \tfrac{1}{2})(\bar{\rho} - \rho_c) + \frac{\omega_l a^2 \rho_c}{2} < \bar{\lambda} < l(\bar{\rho} - \rho_c) + \frac{\omega_l a^2 \rho_c}{2}$$

then

$$\omega(L) \to \omega_l \qquad \tilde{l}_1(\omega(L)) \to l$$

$$\tilde{l}_2(\omega(L)) \to l - 1 \qquad c_L(\omega, \mu) \to l(\bar{\rho} - \rho_c) + \frac{\omega_l a^2 \rho_c}{2} - \bar{\lambda}.$$

(iii) If, for some $l \geqslant 1$,

$$(l - 1)(\bar{\rho} - \rho_c) + \frac{\omega_l a^2 \rho_c}{2} \leqslant \bar{\lambda} < (l - \tfrac{1}{2})(\bar{\rho} - \rho_c) + \frac{\omega_l a^2 \rho_c}{2}$$

then

$$\omega(L) \to \omega_l \qquad \tilde{l}_1(\omega(L)) \to l - 1$$

$$\tilde{l}_2(\omega(L)) \to l \qquad c_L(\omega, \mu) \to -(l - 1)(\bar{\rho} - \rho_c) - \frac{\omega_l a^2 \rho_c}{2} + \bar{\lambda}.$$

(iv) If, for some $l \geqslant 1$,

$$\bar{\lambda} = (l - \tfrac{1}{2})(\bar{\rho} - \rho_c) + \frac{\omega_l a^2 \rho_c}{2}$$

then

$$\omega(L) \to \omega_l$$

$$c_L(\omega, \mu) \to \tfrac{1}{2}(\bar{\rho} - \rho_c).$$

It is clear that for $\bar{\rho} \leqslant \rho_c$ there is no condensation. If $\bar{\rho} > \rho_c$, in case (i), there is condensation only in level 1, as in the ordinary free boson gas. In cases (ii)–(iv), there is condensation in both levels 1 and 2. This is because of the degeneracy of the lowest energy level when $\omega = \omega_l$. In order to compute the extremal states, we add one field in case (i) and two fields in case (ii). We limit ourselves to the explicit presentation of this latter case, because all other cases can be treated in the same way.

Lemma 1. For $f, g \in \mathcal{H}_L$, the equilibrium state described by the Hamiltonian

$$H_L^\epsilon = H_L + \sqrt{L^3}\epsilon_1(a_{1,L}^+ + a_{1,L}) + \sqrt{L^3}\epsilon_2(a_{2,L}^+ + a_{2,L}) \tag{15}$$

is a quasi-free state [11] with the following one- and two-point functions

$$\begin{aligned} \eta_L^\epsilon(a_{1,L}^+) &= -\frac{\epsilon_1\sqrt{L^3}}{E_{1,L} - \Omega_L l_1 - \mu_L} \\ \eta_L^\epsilon(a_{2,L}^+) &= -\frac{\epsilon_2\sqrt{L^3}}{E_{2,L} - \Omega_L l_2 - \mu_L} \\ \eta_L^\epsilon(a_{k,L}^+) &= 0 \qquad \text{for } k > 2 \end{aligned} \tag{16}$$

and

$$\eta_L^\epsilon(a(f)a^+(g)) = \sum_{k=1}^{\infty} \frac{\bar{\hat{f}}_{k,L} g_{k,L}}{1 - \mathrm{e}^{-\beta(E_{k,L} - \Omega_L l_k - \mu_L)}} + \sum_{i=1}^{2} \frac{|\epsilon_i|^2}{(E_{i,L} - \Omega_L l_i - \mu_L)^2} \bar{\hat{f}}_{i,L} g_{i,L}$$

where $f_{k,L} = \langle \phi_{k,L}, f \rangle$.

Proof. This follows straightforwardly from the KMS condition (2). □

For the system with external fields, consider again constraint equations (8) and (9). The chemical potential μ_L now depends on these fields. This is expressed through the notation μ_L^ϵ. The same holds, of course, for the angular velocity. With this in mind, one obtains the following property concerning the joint limit $L \to \infty$, $\epsilon_1 \to 0$, $\epsilon_2 \to 0$.

Lemma 2. Following the scheme of case (ii) from theorem 2, one obtains

(i) $$\lim_{\epsilon\to 0} \lim_{L\to\infty} (E_{1,L} - \Omega_L^\epsilon l_1 - \mu_L^\epsilon) = 0 \tag{17}$$

(ii) $$\lim_{\epsilon\to 0} \lim_{L\to\infty} (E_{2,L} - \Omega_L^\epsilon l_2 - \mu_L^\epsilon) = 0 \tag{18}$$

(iii) $$\lim_{\epsilon\to 0} \lim_{L\to\infty} \frac{|\epsilon_1|^2}{(E_{1,L} - \Omega_L^\epsilon l_1 - \mu_L^\epsilon)^2} = \bar{\lambda} - \frac{\omega_l a^2 \rho_c}{2} + (1 - l)(\bar{\rho} - \rho_c) \tag{19}$$

(iv) $$\lim_{\epsilon\to 0} \lim_{L\to\infty} \frac{|\epsilon_2|^2}{(E_{2,L} - \Omega_L^\epsilon l_2 - \mu_L^\epsilon)^2} = -\bar{\lambda} + \frac{\omega_l a^2 \rho_c}{2} + l(\bar{\rho} - \rho_c) \tag{20}$$

where ω_l and l are as before.

Proof. Relations (i) and (ii) are trivial, (iii) and (iv) become clear if one rewrites constraints (8) and (9) as

$$\bar{\rho} - \rho_c = \lim_{\epsilon\to 0} \lim_{L\to\infty} \frac{|\epsilon_1|^2}{(E_{1,L} - \Omega_L^\epsilon l_1 - \mu_L^\epsilon)^2} + \lim_{\epsilon\to 0} \lim_{L\to\infty} \frac{|\epsilon_2|^2}{(E_{2,L} - \Omega_L^\epsilon l_2 - \mu_L^\epsilon)^2}$$

$$\bar{\lambda} - \tfrac{1}{2}\omega_l a^2 \rho_c = \lim_{\epsilon\to 0} \lim_{L\to\infty} \frac{l_1|\epsilon_1|^2}{(E_{1,L} - \Omega_L^\epsilon l_1 - \mu_L^\epsilon)^2} + \lim_{\epsilon\to 0} \lim_{L\to\infty} \frac{l_2|\epsilon_2|^2}{(E_{2,L} - \Omega_L^\epsilon l_2 - \mu_L^\epsilon)^2}.$$

□

Introduce the notation

$$j_{k,L} = l_k n_{k,L}$$

$$\xi_{k,L} = E_{k,L} - \Omega_L l_k - \mu_L \qquad \text{if } k > 2$$

$$\xi^{\epsilon}_{i,L} = E_{i,L} - \Omega^{\epsilon}_L l_i - \mu^{\epsilon}_L \qquad \text{if } i = 1, 2,$$

$$\sigma^{\epsilon_k}(k, L) = \frac{\exp(-\beta(\xi^{\epsilon}_{k,L} n_{k,L} + \sqrt{L^3}\epsilon_k(a_{k,L} + a^+_{k,L})))}{\mathrm{Tr}\exp(-\beta(\xi^{\epsilon}_{k,L} n_{k,L} + \sqrt{L^3}\epsilon_k(a_{k,L} + a^+_{k,L})))}$$

$$\sigma(k, L) = \frac{\exp(-\beta(\xi_{k,L} n_{k,L}))}{\mathrm{Tr}\exp(-\beta(\xi_{k,L} n_{k,L}))}$$

$$\rho_{k,L} = (\exp(\beta\xi_{k,L}) - 1)^{-1}.$$

Theorem 3. If $\bar{\rho} > \rho_c$, then the extremal equilibrium states are given by the following generating functional:

$$\lim_{\epsilon\to 0}\lim_{L\to\infty} \eta_L(W(f)) = \exp -\left(\frac{\mathrm{i}}{\sqrt{2}}\delta_{l,1}\sqrt{l(\bar{\rho} - \rho_c) - \bar{\lambda} + \frac{\omega_l a^2 \rho_c}{2}}(\hat{f}(0) + \bar{\hat{f}}(0))\right)$$
$$\times \exp(-\tfrac{1}{4} \parallel f \parallel^2) \exp(-\tfrac{1}{2}\mathcal{A}_{\beta,\bar{\rho},\bar{\lambda}}(f, f))$$

for all $f \in \mathcal{D}$.

Proof. The explicit proof is given only for case (ii) of theorem 2. By expanding function f in the basis $\{\phi_{k,L} : k = 1, 2, \ldots\}$ and using the fact that the different k-modes are orthogonal, or

$$\mathcal{A}_L \cong \bigotimes_k \mathcal{A}^k_L \tag{21}$$

where the $\mathcal{A}^k_L$ are generated by $\{a_{k,L}, a^+_{k,L}, \mathbb{1}\}$, one has the following expression for the generating functional:

$$\lim_{\epsilon\to 0}\lim_{L\to\infty} \prod_{j=1}^{2} \mathrm{Tr}\,\sigma^{\epsilon_j}(j, L)\exp\left(\frac{\mathrm{i}}{\sqrt{2}}(\overline{\langle\phi_{j,L}, f\rangle} a_{j,L} + \langle\phi_{j,L}, f\rangle a^+_{j,L})\right)$$
$$\times \prod_{k=3}^{\infty} \mathrm{Tr}\,\sigma(k, L)\exp\left(\frac{\mathrm{i}}{\sqrt{2}}(\overline{\langle\phi_{k,L}, f\rangle} a_{k,L} + \langle\phi_{k,L}, f\rangle a^+_{k,L})\right).$$

Applying the translation automorphisms

$$\begin{aligned} a_{i,L} &\longrightarrow a_{i,L} + \alpha_i \\ a^+_{i,L} &\longrightarrow a^+_{i,L} + \bar{\alpha}_i \end{aligned} \qquad i = 1, 2$$

where

$$\alpha_i = \frac{\sqrt{L^3}\epsilon_i}{E_{i,L} - \Omega^{\epsilon}_L l_i - \mu^{\epsilon}_L}$$

one obtains

$$\lim_{\epsilon\to 0}\lim_{L\to\infty}\exp\frac{\mathrm{i}}{\sqrt{2}}\left(\sum_{j=1}^{2}\left(\overline{\langle\phi_{j,L},f\rangle}+\langle\phi_{j,L},f\rangle\right)\frac{\epsilon_j}{E_{j,L}-\Omega_L^{\epsilon}l_j-\mu_L^{\epsilon}}\right)$$
$$\times\prod_{k=1}^{\infty}\mathrm{Tr}\,\sigma(k,L)\exp\frac{\mathrm{i}}{2}(\overline{\langle\phi_{k,L},f\rangle}a_{k,L}+\langle\phi_{k,L},f\rangle a_{k,L}^{+}).$$

In the limit $L\to\infty$, the product over k becomes

$$\exp -(\tfrac{1}{4}\|f\|^2+\tfrac{1}{2}\mathcal{A}_{\beta,\bar{\rho},\bar{\lambda}}(f,f)).$$

Using

$$\lim_{L\to\infty}\langle\phi_{1,L},f\rangle=\lim_{L\to\infty}K_{l,1}\int_{\Lambda_L}r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z\,f(r,\theta,z)J_{|l|}\left(\frac{r_{l,n}a^{-1}r}{L}\right)\mathrm{e}^{\mathrm{i}l\theta}\cos\left(\frac{\pi^2a^2mz}{L}+\frac{\pi m}{2}\right)$$
$$=\delta_{l,0}\hat{f}(o)$$
$$\lim_{L\to\infty}\langle\phi_{2,L},f\rangle=\delta_{l,1}\hat{f}(0)$$

where $\hat{f}$ is the Fourier transform of f and formulae (19) and (20), the terms with the fields yield (because $l\geqslant 1$) (ii) from theorem 2:

$$\exp\frac{\mathrm{i}}{\sqrt{2}}\delta_{l,1}\sqrt{l(\bar{\rho}-\rho_c)-\bar{\lambda}+(\omega_l a^2\rho_c)/2}(\hat{f}(0)+\bar{\hat{f}}(0))\exp -(\tfrac{1}{4}\|f\|^2+\tfrac{1}{2}\mathcal{A}_{\beta,\bar{\rho},\bar{\lambda}}(f,f)).$$

One can prove analogously cases (i), (iii) and (iv) of theorem 2. Collating this, the generating functional for $\bar{\rho}\geqslant\rho_c$ becomes

$$\lim_{\epsilon\to 0}\lim_{L\to\infty}\eta_L(W(f))=\exp -\frac{\mathrm{i}}{\sqrt{2}}\sqrt{\bar{\rho}-\rho_c}\sqrt{G\left(\frac{\bar{\lambda}-(\omega_1a^2\rho_c)/2}{(\bar{\rho}-\rho_c)}\right)}(\hat{f}(0)+\bar{\hat{f}}(0))$$
$$\times\exp -(\tfrac{1}{4}\ \|\ f\ \|^2+\tfrac{1}{2}\mathcal{A}_{\beta,\bar{\rho},\bar{\lambda}}(f,f)). \qquad\square$$

Notice that if $G=0$, there is no contribution from the condensate to the generating functional. This is due to the fact that in the formulation of the theorem, one calculates with functions living on a compact support. If, on the other hand, one calculates with arbitrary functions $f\in\mathcal{H}_L$. with an expanding support and then takes the limit $L\to\infty$ as in lemma 2, then the condensate does appear in the expressions. The reason is that if the angular momentum is large enough (i.e. $l\geqslant 2$), the condensate moves to the boundary under the influence of centrifugal forces. This can be made more explicit, e.g. in case (i) of theorem 1, the angular momentum is then given by the formula [3]

$$\bar{\lambda}=\omega_\infty a^2(\bar{\rho}-\rho_c)+\tfrac{1}{2}\omega_\infty a^2\rho_c.$$

The first term represents the angular momentum of a ring with radius a, rotating with an angular velocity ω_∞ and a particle density $\bar{\rho}-\rho_c$, while the second term is the angular momentum of a cylinder of fluid particles with radius a, rotating at the same speed, with a particle density ρ_c.

3. Angular momentum distributions

In this section, we look for the distribution of the average angular momentum $\bar{\lambda}$ and the average density $\bar{\rho}$. The results give information on whether the limit state $\eta_\infty = \lim_{L\to\infty} \eta_L$ is an extremal equilibrium state or not. In fact, we calculate the distribution function of $\bar{\lambda}$.

Theorem 4. Consider system (1) at $\bar{\rho} \leqslant \rho_c$, then

$$\lim_{L\to\infty} \eta_L \left(\exp\left(\frac{is}{L^3} J_L \right) \right) = \exp(is\bar{\lambda}). \tag{22}$$

Proof. By (7)

$$\bar{\lambda} = \frac{1}{L^3} \sum_{k=1}^{\infty} l_k \eta_L(n_{k,L})$$

implying that $\eta_L(J_L)$ is of order $O(L^3)$. The left-hand side of (22) becomes

$$\begin{aligned} \lim_{L\to\infty} \mathrm{Tr} \prod_{k=1}^{\infty} \sigma(k, L) \exp\left(\frac{is}{L^3} J_L \right) &= \lim_{L\to\infty} \prod_{k=1}^{\infty} \frac{1 - \exp(-\beta \xi_{k,L})}{1 - \exp(-\beta \xi_{k,L} + (is/L^3) l_k)} \\ &= \lim_{L\to\infty} \exp - \sum_{k=1}^{\infty} \ln(1 - \epsilon_{k,L}) \end{aligned} \tag{23}$$

where $\epsilon_{k,L} = \rho_{k,L}(e^{isL^{-3}l_k} - 1)$. Next, one uses the bounds

$$\begin{aligned} &|\ln(1+z) - z| \leqslant |z|^2 \qquad \text{for } |z| \leqslant \tfrac{1}{2} \\ &|e^{i\alpha} - 1|^2 \leqslant \alpha^2 \\ &\rho_{k,L}^2 \leqslant \rho_{k,L} \qquad \text{for all } k \geqslant 1 \end{aligned} \tag{24}$$

to obtain

$$\sum_{k=1}^{\infty} |\epsilon_{k,L}|^2 \leqslant s^2 \frac{1}{L} \left(\frac{1}{L^3} \sum_{k=1}^{\infty} \rho_{k,L} (l_k L^{-1})^{-2} \right). \tag{25}$$

The expression in the bracket is convergent to a finite integral. The supplementary L^{-1} factor makes the whole expression tend to zero, by the dominated convergence theorem. By expansion of the exponential $\exp(isL^{-3}l_k)$ and using the same argument as above, one obtains the result: $\exp(is\bar{\lambda})$. □

Analogously, one shows that the distribution of the average density equals

$$\lim_{L\to\infty} \eta_L \left(\exp\left(\frac{is}{L^3} N_L \right) \right) = \exp(is\bar{\rho})$$

in the case $\bar{\rho} \leqslant \rho_c$. Note that this agrees with the result for the free boson gas (see, e.g. [12]).

Let us now consider the case $\bar{\rho} > \rho_c$, and calculate again the distribution in the state determined by Hamiltonian (1). The expectation value of the distribution function will *not* be a δ-distribution.

Theorem 5. If $\bar{\rho} > \rho_c$, for case (i) of theorem 2, one obtains

$$\lim_{L\to\infty} \eta_L \left(\exp\left(\frac{is}{L^3} J_L \right) \right) = \frac{\exp(is\frac{1}{2}\omega_\infty a^2 \rho_c)}{1 - is(\bar{\rho} - \rho_c) l_1(\omega_\infty)}. \tag{26}$$

Proof. We remark that, in this case, there is condensate only in the first level, i.e.

$$\lim_{L\to\infty} \frac{1}{L^3} \rho_{1,L} = \bar{\rho} - \rho_c \tag{27}$$

while

$$\lim_{L\to\infty} \frac{1}{L^3} \rho_{k,L} = 0 \qquad \text{for } k \geqslant 2. \tag{28}$$

Using the same technique as in the proof of theorem (4) for equation (23), one obtains

$$\lim_{L\to\infty} \eta_L(\exp(isL^{-3} J_L)) = \lim_{L\to\infty} \frac{1 - \exp(-\beta\xi_{1,L})}{1 - \exp(-\beta\xi_{1,L} + isL^{-3} l_1)} \exp\left(\sum_{k=1}^{\infty} \ln(1 - \epsilon_{k,L}) \right). \tag{29}$$

Using (27) and (28), the first factor becomes, in the limit $L \to \infty$, equal to

$$\frac{1}{1 - isl_1(\bar{\rho} - \rho_c)}.$$

The factor $\exp(-\sum_{k=2}^{\infty} \ln(1 - \epsilon_{k,L}))$ converges, similarly as in the proof of theorem 4, to $\exp(is\frac{1}{2}\omega_\infty a^2 \rho_c)$. This concludes the proof. □

This is clearly not the Fourier transform of a δ-distribution, and again this result is similar to the result for the free boson gas [12]. Similarly, one computes the distribution in the cases when both levels show condensation. The result is then

$$\exp(is\lambda_c) \prod_{i=1}^{2} (1 - is(l_i(\omega_i)(\bar{\rho} - \rho_c) - (\bar{\lambda} - \tfrac{1}{2}\omega_i a^2 \rho_c)))^{-1}.$$

We remark that for theorem 5, we do not have a convenient law of large numbers for the angular momentum in the equilibrium state η_L. Therefore, we now calculate the distribution function of the angular momentum operator for $\bar{\rho} > \rho_c$ in the state η_L^ϵ with external fields (see lemma 1). We restrict ourselves to case (ii) of theorem 2. We take the fields ϵ_i to be volume dependent in the following particular way:

$$\epsilon_i = \frac{C_i}{L^{3\alpha}} \qquad i = 1, 2 \qquad \text{with } \alpha > 0.$$

The external field plays the role of a boundary condition, it vanishes with increasing volumes.

Lemma 3. For volume-dependent fields the energy gaps have the following volume dependence (for $i = 1, 2$):

$$0 < \alpha < 1 : (E_{i,L} - \Omega_L l_i - \mu_L) \sim O(L^{-3\alpha})$$

$$1 \leqslant \alpha : (E_{i,L} - \Omega_L l_i - \mu_L) \sim O(L^{-3}).$$

Proof. For $0 < \alpha < 1$, by lemma 1, if one has two-fold degeneracy of the grand canonical-state spectrum of $H_L - \mu_L N_L$, then

$$\bar{\rho} = \lim_{L\to\infty} \left\{ \frac{C_1^2}{L^{6\alpha}(E_{1,L} - \Omega_L l_1 - \mu_L)^2} + \frac{C_2^2}{L^{6\alpha}(E_{1,L} - \Omega_L l_1 - \mu_L)^2} + \frac{2}{L^3}\rho_{1,L} + \rho_c \right\}.$$

Hence,

$$\bar{\rho} - \rho_c = \lim_{L\to\infty} \frac{1}{L^{3\alpha}(E_{1,L} - \Omega_L l_1 - \mu_L)} \lim_{L\to\infty} \left\{ \frac{C_1^2}{L^{3\alpha}(E_{1,L} - \Omega_L l_1 - \mu_L)} + \frac{C_2^2}{L^{3\alpha}(E_{1,L} - \Omega_L l_1 - \mu_L)} + \frac{2}{L^3}(E_{1,L} - \Omega_L l_1 - \mu_L) L^{3\alpha} \rho_{1,L} \right\}.$$

The statement of the theorem follows immediately from the fact that the average density is kept fixed. Note that the contribution to $\bar{\rho} - \rho_c$ comes from the first two terms. If $\alpha \geqslant 1$, only the last term contributes and, therefore, $(E_{i,L} - \Omega_L l_i - \mu_L) \sim O(L^{-3})$. □

In order to calculate the distribution function, we need the following lemma.

Lemma 4. For $k = 1, 2$

$$\mathrm{Tr}\left(\sigma^{\epsilon_k}(k, L) \exp\left(\frac{\mathrm{i}s}{L^3} j_{k,L}\right)\right) = \exp(\mathrm{i}s l_k f(k))$$

where

$$f(1) = \bar{\lambda} - \tfrac{1}{2}\omega_l a^2 \rho_c - (l-1)(\bar{\rho} - \rho_c)$$

$$f(2) = l(\bar{\rho} - \rho_c) + \tfrac{1}{2}\omega_l a^2 \rho_c - \bar{\lambda}$$

and $\epsilon_k = C_k / L^{3\alpha}$, $0 < \alpha < 1$.

Proof. In order to compute the trace, apply again the translation automorphism, as in the proof of theorem 3 (using the same definition for α_k), then one obtains for the trace

$$\exp\left(\frac{\mathrm{i}s l_k}{L^3}|\alpha_k|^2\right) \mathrm{Tr}\left(\sigma(k, L) \exp\left(\frac{\mathrm{i}s l_k}{L^3}(\alpha_k a_{k,L}^+ + \bar{\alpha}_k a_{k,L} + n_{k,L})\right)\right). \tag{30}$$

Defining

$$U(t) = \exp\left(-\frac{\mathrm{i}s}{L^3} j_{k,L} t\right) \exp \frac{\mathrm{i}s l_k}{L^3}(\alpha_k a_{k,L}^+ + \bar{\alpha}_k a_{k,L} + n_{k,L})t$$

one has the following differential equation for $U(t)$:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \left[\exp\left(-\frac{\mathrm{i}s}{L^3} j_{k,L} t\right)\right] \frac{\mathrm{i}s l_k}{L^3} (\alpha_k a_{k,L}^+ + \bar{\alpha}_k a_{k,L}) \left[\exp\left(\frac{\mathrm{i}s}{L^3} j_{k,L} t\right) U(t)\right].$$

The solution of this equation, evaluated at $t = 1$, is given by

$$U(1) = \exp\left(\alpha_k \left(\exp\left(\frac{\mathrm{i}s}{L^3} l_k\right) - 1\right) a_{k,L}^+ - \bar{\alpha}_k \left(\exp\left(-\frac{\mathrm{i}s}{L^3} l_k\right) - 1\right) a_{k,L} + \frac{\mathrm{i}s l_k}{2L^3}\left(\cos\left(\frac{\mathrm{i}s l_k}{L^3}\right) - 1\right)\right).$$

Using in (30)

$$\mathbb{1} = \exp\left(\frac{\mathrm{i}s}{L^3} j_{k,L}\right) \exp\left(-\frac{\mathrm{i}s}{L^3} j_{k,L}\right)$$

one obtains for the trace

$$\exp\left(\frac{\mathrm{i}s}{L^3} l_k \frac{L^3 |\epsilon_k|^2}{(-\beta\xi_{k,L})^2}\right) \frac{1 - \exp(-\beta\xi_{k,L})}{1 - \exp(-\beta\xi_{k,L} + (\mathrm{i}s l_k / L^3))} \exp\left\{-\left|\alpha_k \left(\exp\left(\frac{\mathrm{i}s l_k}{L^3}\right) - 1\right)\right|^2 \times \left(\frac{1}{2} + \frac{1}{\exp(-(-\beta\xi_{k,L} + (\mathrm{i}s l_k/L^3))) - 1}\right) + \frac{\mathrm{i}s l_k}{2L^3}\left(\cos\left(\frac{\mathrm{i}s l_k}{L^3}\right) - 1\right)\right\}.$$

After taking the limit $L \to \infty$, one obtains (using lemma 3) lemma 4. □

Theorem 6. If $\bar{\rho} \geqslant \rho_c$, then, for the state η_L^ϵ (lemma 1),

$$\lim_{L\to\infty} \eta_L^\epsilon \left(\exp\left(\frac{\mathrm{i}s}{L^3} J_L\right)\right) = \exp(\mathrm{i}s\bar{\lambda}). \tag{31}$$

Proof. We write down the proof explicitly only for case (ii) of theorem 2.

$$\lim_{L\to\infty} \eta_L^\epsilon \exp\left(\frac{\mathrm{i}s}{L^3} J_L\right) = \lim_{L\to\infty} \prod_{j=1}^{2} \mathrm{Tr}\left(\sigma^{\epsilon_j}(j, L) \exp\left(\frac{\mathrm{i}s}{L^3} j_{j,L}\right)\right) \times \prod_{k=3}^{\infty} \mathrm{Tr}\left(\sigma(k, L) \exp\left(\frac{\mathrm{i}s}{L^3} \sum_{k=3}^{\infty} j_{k,L}\right)\right).$$

Using the previous lemma yields

$$\lim_{L\to\infty} \eta_L^\epsilon \left(\exp\left(\frac{\mathrm{i}s}{L^3} J_L\right)\right) = \exp(\mathrm{i}sl(\bar{\lambda} - \tfrac{1}{2}\omega_l a^2 \rho_c - (l-1)(\bar{\rho} - \rho_c))) \times \exp(\mathrm{i}s[(l-1)(l(\bar{\rho} - \rho_c) + \tfrac{1}{2}\omega_l a^2 \rho_c - \bar{\lambda}) + \tfrac{1}{2}\omega_\infty a^2 \rho_c] = \exp(\mathrm{i}s\bar{\lambda}).$$ □

This theorem constitutes our most striking result of this section. One remarks that even at high densities $\bar{\rho} > \rho_c$, the distribution of the average angular momentum in the extremal equilibrium states is still a point distribution. This indicates that our technique of working with a two-mode external field lifts the degeneracy of the ground-state levels. It indicates also that the extremal or ergodic equilibrium states, computed in theorem 3, are the grand canonical states in which one should compute the angular momentum fluctuations. This is achieved in the next section.

4. Angular momentum fluctuations

In this section, the aim is to find the critical exponents of the angular momentum susceptibility. We calculate the fluctuation of the angular momentum in a special way; we simultaneously take the limit, tending to infinity, of the size of the system together with the number of random variables. To be precise, let $\{\omega_L\}_L$ be a sequence of finite volume KMS states, such that $\lim_{L\to\infty}\omega_L = \omega_\infty$ is an ergodic equilibrium state. Then, for any local observable A, we look for the parameter $\delta \in]-\frac{1}{2}, \frac{1}{2}[$ for which the following variance is non-trivial:

$$0 < \lim_{L\to\infty} \eta_L(F_\delta^2(A)) \equiv \lim_{L\to\infty} \frac{1}{L^{3+6\delta}} \eta_L((A - \eta_L(A))^2) < \infty. \tag{32}$$

The critical exponent δ indicates at which level the fluctuations appear, i.e. δ is a measure for the deviation from the standard square root of the mathematically-normal fluctuation. For translationally-invariant operators, $\delta = 0$ indicates the normal situation, the fluctuations are then called *normal*. If $\delta > 0$, one speaks about *abnormal* fluctuations; if $\delta < 0$ then $F_\delta(A)$ is called a *subnormal (squeezed) critical* fluctuation. In our case, we have to take for A the angular momentum J, i.e.

$$F_\delta(J) = \lim_{L\to\infty} \frac{1}{L^{3(\frac{1}{2}+\delta)}} (J_L - \bar{\lambda} L^3).$$

As we shall show, the *physically normal* situation for the angular momentum J is $\delta = \frac{1}{3}$. This is because of the fact that the angular momentum is not a translationally-invariant operator. The result agrees with an equivalent result for the interacting Bose gas under suitable cluster conditions, e.g. at very high temperatures [13]. The $\delta = \frac{1}{3}$ is also related to the fact that the angular momentum fluctuations are connected to the moment of inertia of the system in the following way. The moment of inertia is defined as the derivative of the angular momentum with respect to the circular velocity:

$$I = \left.\frac{\partial \langle J\rangle}{\partial \Omega}\right|_{\Omega=0}$$

where, of course,

$$\langle J\rangle = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta(H-\Omega J)} J)}{\mathrm{Tr}(\mathrm{e}^{-\beta(H-\Omega J)})}.$$

Clearly,

$$I = \beta(\langle J^2\rangle - \langle J\rangle^2|_{\Omega=0}).$$

Up to a volume-dependent factor, this is the angular momentum fluctuation variance and, in the thermodynamic limit, one indeed obtains $\lim_{L\to\infty}\Omega_L = 0$. This is the heuristic argument for why the variance of the angular momentum fluctuation is proportional to the moment of inertia.

First we compute the value of δ at low densities.

Theorem 7. If $\bar{\rho} < \rho_c$, then $\delta = \frac{1}{3}$.

Proof. Compute using lemma 1

$$\eta_L(F_\delta^2(J)) = \frac{1}{L^{3(1+2\delta)}}\left\{\sum_{k=1}^{\infty} l_k^2\rho_{k,L}^2 + l_k^2\rho_{k,L}\right\}.$$

Because $r_{l,n} \geqslant |l| + n - 2$, for $|l| + n \geqslant 2$ [3], one obtains the following upper bound for this sum (we omit the terms $|l| + n \leqslant 2$):

$$\frac{1}{L^{3(1+2\delta)}}\sum_{|l|\geqslant 2}^{\infty}\sum_{n=0,m=0}^{\infty}\left\{l^2\rho_{n,l,m,L}^2 + l^2\rho_{n,l,m,L}\right\}$$

where $k = (n, l, m)$ and

$$\hat{\xi}(n, l, m, L) = \frac{1}{L^2}\left(\frac{\pi^4 a^4 m^2}{2} + \frac{(l + 2n - 4)^2}{8a^2} - \frac{\omega(l + 1)}{2} - \mu_L\right).$$

Hence, up to some constant terms which do not diverge because of the absence of a condensate, the fluctuation is bounded by an expression of the following form:

$$\frac{1}{L^{3(1+2\delta)}}\sum_{|l|\geqslant 0}^{\infty}\sum_{n,m=0}^{\infty} l^2 f\left(\frac{l}{L}, \frac{m}{L}, \frac{n}{L}\right).$$

Putting $\delta = \frac{1}{3}$, one obtains

$$\frac{1}{L^3}\sum_{|l|\geqslant 0}^{\infty}\sum_{n,m=0}^{\infty}\frac{l^2}{L^2} f\left(\frac{l}{L}, \frac{m}{L}, \frac{n}{L}\right) = \int_{-\infty}^{\infty}\int_0^{\infty} \mathrm{d}l\,\mathrm{d}m\,\mathrm{d}n\, f\left(\frac{[lL]}{L}, \frac{[nL]}{L}, \frac{[mL]}{L}\right)\frac{[l^2L^2]}{L^2}$$

where $[x]$ denotes the integral part of the number $x \in \mathbb{R}$. By the dominated convergence theorem, this sum converges to an integral expression which is non-trivial. This proves that δ should be equal to $\frac{1}{3}$ in order to have a finite non-trivial variance (27) for the explicit case (28). □

Now we turn to the more interesting region of high densities.

Theorem 8. If $\tilde{\rho} > \rho_c$, then one obtains

$$\delta = \max\left\{\tfrac{1}{3}, \tfrac{\alpha}{2}\right\} \qquad \text{if } 0 < \alpha < 1$$

$$\delta = \tfrac{1}{2} \qquad \text{if } \alpha \geqslant 1$$

for the limit extremal equilibrium states $\lim_{L\to\infty}\eta_L^\epsilon$ with $\epsilon_i = C_i/L^{3\alpha}$.

Proof. We consider again explicitly only case (ii) of theorem 2. Using lemma 1, one obtains

$$\begin{aligned}\eta_L^\epsilon(F_\delta^2(J)) = {} & \frac{1}{L^{3(1+2\delta)}}\sum_{k=1}^{\infty}\left\{l_k^2\rho_{k,L}^2 + l_k^2\rho_{k,L}\right\} + \frac{2l_1^2|\epsilon_1|^2L^3}{(E_{1,L} - \Omega_L l_1 - \mu_L)^2}\rho_{1,L}\exp(\beta\xi_{1,L}) \\ & + \frac{2l_2^2|\epsilon_2|^2L^3}{(E_{1,L} - \Omega_L l_1 - \mu_L)^2}\rho_{2,L}\exp(\beta\xi_{2,L}) \\ & + \frac{l_1^2|\epsilon_1|^2L^3}{(E_{1,L} - \Omega_L l_1 - \mu_L)^2} + \frac{l_2^2|\epsilon_2|^2L^3}{(E_{1,L} - \Omega_L l_1 - \mu_L)^2}.\end{aligned}$$

Suppose first that $0 < \alpha < 1$. The expression splits into two parts. The terms with $k \geqslant 3$ behave as $O(L^5)$, as shown in the previous theorem. The first two terms, according to lemma 3, behave as $O(L^{3+3\alpha})$. Combining these two results, we see that $\delta = \max\{\frac{1}{3}, \frac{\alpha}{2}\}$ satisfies criterion (32).

If $\alpha \geqslant 1$, we know from lemma 3 that the first two terms behave as $O(L^6)$, hence $\delta = \frac{1}{2}$. □

From the proof of this theorem, it is clear that if $\alpha > \frac{2}{3}$, the contribution to the angular momentum susceptibility comes only from the levels which show condensate. It is in this region that the effect of the external field is strong enough to influence the fluctuations; $\alpha = \frac{2}{3}$ is the point where the moment of inertia of the system is completely determined by the condensate sitting at the boundary of the cylinder.

The result of this theorem is somewhat surprising. One understands that if the external field drops off very quickly, i.e. for $\alpha \geqslant 1$, then the system behaves as if there is no field and $\delta = \frac{1}{2}$ which corresponds to the case of a mixed state, i.e. a convex combination of extremal equilibrium states. In the free boson gas, we are in this situation. Below the transition point, for high densities, the equilibrium state is the integral over the equilibrium states with fixed gauge. Hence, for $\alpha \geqslant 1$, one expects the same properties as for the full Gibbs state.

When α is smaller, one expects to look at the intrinsic properties of the equilibrium states which are situated in the gauge-breaking extremal states. Also, the study of the deviation from normality of the angular momentum fluctuation should refer to these gauge-breaking states. This is exactly what we expect to find for $\alpha \leqslant 1$. The theorem states that the parameter δ depends on the boundary condition, i.e. on α. This phenomenon has already been found in other models [5, 6, 14] as well as quantum and classical models. Here this effect is only seen in the range $\frac{2}{3} < \alpha < 1$. However, if $\alpha < \frac{2}{3}$, or the external field vanishes very slowly, then the field forces the system into an extremal phase, with an angular momentum distribution which is Gaussian (see section 5) and physically normal or classical, i.e. the superfluid region does not show any peculiar quantum effect, i.e. $\delta = \frac{1}{3}$, the same value as in the low density region $\bar{\rho} < \rho_c$ (see theorem 7). Hence, the *non*-classical behaviour appears only if $\frac{2}{3} < \alpha < 1$ and it is determined by the boundary condition. This is surprising for a model describing superfluidity.

5. Fluctuation distributions

In this final section, we look for the distributions of the fluctuations of the angular momentum and prove that they are Gausssian in all circumstances, i.e. for high as well as for low densities, and for all values of the average angular momentum.

Theorem 9. If $\bar{\rho} < \rho_c$, then $\delta = \frac{1}{3}$ and

$$\lim_{L\to\infty} \eta_L \left(\exp\left(\frac{is}{L^{3(\frac{1}{2}+\frac{1}{3})}} (J_L - \bar{\lambda} L^3) \right) \right) = \exp\left(-\frac{s^2}{2} \eta_\infty (F_\delta^2(J)) \right).$$

Proof. Using the same technique as in the proof of theorem 4, one obtains

$$\lim_{L\to\infty} \eta_L \Bigg(\exp \Bigg\{ -\sum_{k=1}^{\infty} (\ln(1 - \exp(-\beta\xi(k, L))) - \ln(1 - \exp(-\beta\xi(k, L) + isl_k L^{-3(\frac{1}{2}+\frac{1}{3})}))) - isl_k L^{-3(\frac{1}{2}+\frac{1}{3})} \Bigg\} \Bigg).$$

Again, on the basis of the bounds used there, the linear term disappears due to constraint (9) and only the quadratic term survives. One obtains

$$\lim_{L\to\infty} \tfrac{1}{2}(is)^2 L^{-6(\frac{1}{2}+\frac{1}{3})} \sum_{k=1}^{\infty} l_k^2 (\rho_{k,L}^2 + \rho_{k,L}) = -\tfrac{1}{2} s^2 \eta_\infty (F_\delta^2). \qquad \square$$

Theorem 10. If $\bar{\rho} > \rho_c$ and $0 < \alpha < 1$ then, with δ as in theorem 8, one again obtains

$$\lim_{L\to\infty} \eta_L^{\epsilon}\left(\exp\left(\frac{is}{L^{3(\frac{1}{2}+\delta)}}(J_L - \bar{\lambda}L^3)\right)\right) = \exp\left(-\frac{s^2}{2}\eta_\infty^{\epsilon}(F_\delta^2(J))\right).$$

Proof. We perform the calculation explicitly for case (ii) of theorem 2. We remind ourselves that

$$\epsilon_i = \frac{C_i}{L^{3\alpha}} \qquad \text{for } i = 1, 2.$$

The distribution function equals

$$\lim_{L\to\infty} \exp\left(-\frac{is\bar{\lambda}L^3}{L^{3(\frac{1}{2}+\delta)}}\right)\prod_{k=1}^{2}\mathrm{Tr}(\sigma^{\epsilon_k}(k, L)\exp(isL^{-3(\frac{1}{2}+\delta)}J_L))$$
$$\times\prod_{k=3}^{\infty}\mathrm{Tr}(\sigma(k, L)\exp(-isL^{-3(\frac{1}{2}+\delta)}J_L)).$$

Using the same technique as in lemma 4, the first trace becomes

$$\exp\left(\frac{is}{L^{3(\frac{1}{2}+\delta)}}l_k\frac{L^3|\epsilon_k|^2}{(\beta\xi_{k,L})^2}\right)\frac{1-\exp(-\beta\xi_{k,L})}{1-\exp(-\beta\xi_{k,L}+(isl_k/L^{3(\frac{1}{2}+\delta)}))}$$
$$\times\exp\left(-\left|\alpha_k\left(\exp\left(\frac{isl_k}{L^{3(\frac{1}{2}+\delta)}}\right)-1\right)\right|^2\left(\frac{1}{2}+\frac{1}{\exp(\beta\xi_{k,L}-(isl_k/L^{3(\frac{1}{2}+\delta)})-1}\right)\right.$$
$$\left.+\frac{isl_k}{2L^{3(\frac{1}{2}+\delta)}}\left(\cos\left(\frac{isl_k}{L^{3(\frac{1}{2}+\delta)}}\right)-1\right)\right)$$

where

$$\alpha_k = \frac{\sqrt{L^3}\epsilon_k}{E_{1,L}-\Omega_L l_1-\mu_L}.$$

If $\alpha < \frac{2}{3}$, this expression tends to one in the limit $L\to\infty$, as can be seen by using lemma 3. Hence, in this case, the distribution function equals

$$\exp\left(-\frac{s^2}{2}\eta_\infty(F_\delta^2(J))\right).$$

If, however, $\alpha > \frac{2}{3}$ then, by slightly modifying the proof of lemma 4, one obtains

$$\lim_{L\to\infty}\exp\left(\frac{l_1^2|\epsilon_1|^2L^3\rho_{1,L}^2}{(E_{1,L}-\Omega_L l_1-\mu_L)^2}+\frac{l_2^2|\epsilon_2|^2L^3\rho_{2,L}^2}{(E_{1,L}-\Omega_L l_1-\mu_L)^2}\right).$$

Looking at the proof of theorem 8, one sees that in the limit $L\to\infty$, the exponent exactly equals the contribution to the fluctuation, giving for the distribution

$$\exp\left(\frac{-s^2}{2}\eta_\infty(F_\delta^2(J))\right).$$

□

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